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Sharmin R. Ara
*East West University*

R. Viswanathan
*Southern Illinois University Carbondale*, viswa@engr.siu.edu



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# Decision Combining in Relay Networks

Sharmin R Ara and R. Viswanathan *

Department of Electrical and Electronic Engineering, East West University, Dhaka, Bangladesh,*Department of Electrical & Computer Engineering , Southern Illinois University at Carbondale, USA
E-mail: shra@ewu.edu.bd,viswa@engr.siu.edu

**Abstract - We consider non-coherent detection of *M-ary* FSK modulated signals transmitted over a slow, Rayleigh fading channel in a wireless relay network. The network consists of a single source-destination pair and a number of relays (*L*), which employ cooperative diversity. Performances of a counting rule and square law combiner are studied. We derive closed form expressions for probabilities of error for equal relay channel average SNR. For unequal relay channel SNRs, we resort to Monte Carlo simulations to estimate the error probabilities. We examine different combinations of *M* and *L* for a range of average SNR values. Although the square law combiner outperforms the counting rule for equal and small average SNRs, the loss in performance is not high. Simplicity of counting rule may be advantageous in these cases.**

## I. Introduction

Cooperative diversity is a popular and effective technique to mitigate fading in wireless network. This scheme could be effectively used for cellular, satellite and certain wireless local area network (LAN). Cooperative diversity exploits the broadcast nature and inherent spatial diversity of the channel [1]-[5], i.e., transmitted signal can be received and processed by any number of terminals. Based on relaying procedure, cooperative diversity can be broadly categorized as fixed relaying and adaptive relaying. Amplify and forward (AF) and decode and forward (DF) belong to the first category. In a relaying protocol, communication takes place in two phases. In first phase, the signals are transmitted by the source to the destination in a broadcast manner. All relays and destination receive faded noisy versions of these signals. In second phase, the relays retransmit a processed version of the received signals to the destination, and finally the destination combines the signals received in two phases. A comprehensive analysis of statistical properties of AF relay channels such as auto-correlation, level crossing rate (LCR), and average outage duration (AOD) were studied in [6]. Combining procedures for *M-ary* hypothesis testing with diversity link appeared in [7]. In an attempt to find channel aware processing that minimizes the error probability at destination node, an iterative algorithm was presented to find relay schemes that are at least locally optimum [4].

In this paper we are considering amplify and forward relays for a single source-to-destination (S-D) pair of network. Assuming non-coherent detection [8], our aim is to find a simple procedure to calculate probability of error at destination. Since detection is non-coherent, arriving signal phase information is not needed. We will restrict our work to *M-ary* orthogonal FSK, which is appropriate for both slow and fast fading, though we will assume a sufficiently slow, Rayleigh fading channel. Also, fading processes on the *L*-channels are assumed to be mutually statistically independent.

## II. Counting Rule and Multinomial Distribution

In this section a simple non parametric procedure such as decision combining, as a method of aggregating the information arriving through different channels, is proposed. A single S-D pair with a large number of relays in a wireless network is considered. The relays are of identical capacity and performance. It is assumed that the symbols are sent over a Rayleigh fading channel using non coherent modulation such as $M$ *-ary* FSK; Each channel is assumed to be frequency- nonselective and mutually statistically independent. The signal received is corrupted by additive white Gaussian noise. Hence, for each identical relay channel, the probability of correct symbol reception at D equals $P_C = (1 - P_e)$, where $P_e$ is the probability of symbol error for a Relay link and probability that any one of the symbols, other than the transmitted one, is selected is $P_e/(M-1)$. Given that there are $L$ relay links and given decisions are arrived at by processing the received signals individually, counting the numbers of links that have decided on 1 through $M$, is a sufficient statistic. Calculating the probability of final correct decision (PCS) can be conveniently arrived at by using results from ranking and selection problems in statistics [9]. Considering a multinomial distribution with $M$ cells, where the cell $\pi_i$ has probability $p_i$, for selecting the cell with largest probability $p_{[M]}$ Bechhofer, Elmaghraby and Morse proposed a fixed sample procedure [9].

If $Y_{i,L}$ denotes the number of observations that arise in the cell $\pi_i$, the procedure is given by: Select cell $\pi_i$ for which

$$Y_{i,L} = \max_{1 \le r \le M} Y_{r,L} \quad (1)$$

With the provision that a tie will be broken by randomization, the probability of correct selection (PCS) is given by

$$P_{cs} = \sum_{y_1 \cdot\cdot y_M} \frac{1}{s} \frac{N!}{y_1! \ldots y_M!} p^{y_1}{}_{[1]} \ldots\ldots p_{[M]}{}^{y_M}, \quad (2)$$

$$[\sum_{i=1}^{M} y_i = L,\ y_M \ge y_i, i = 1,\ldots. M-1\ \ ;],$$

where $s$ denotes the number of ties ,

$$p_{[M]} = P_c, \quad (3)$$

$$p_{[i]} = \frac{(1-P_c)}{M-1}, i = 1,2,..M\text{-}1. \quad (4)$$

Now, if we define a parameter to measure the quality of the link such that,

$$\theta = \frac{p_{[M]}}{p_{[i]}}, i = 1,\ldots\ldots M-1 \quad (5)$$

$$\theta = (M-1)Pc/(1-Pc) \quad (6)$$

For a perfect link, $\theta$ tends to infinity, and in the worst case, it approaches 1. For *M-ary* in slow Rayleigh fading channel with non coherent detection, the probability of correct symbol decision in a relay is (pp. 834, [8] )

$$P_c = 1 - \sum_{m=1}^{M-1} \frac{(-1)^{m+1}\begin{pmatrix} M-1 \\ m \end{pmatrix}}{1 + m + m\overline{\gamma_c}} \quad (7)$$

where $\overline{\gamma_c}$ is the average SNR of a relay link. Using (2)-(7), we get the probability of correct selection for different sets of hypothesis ($M$) and relays ($L$) as :

For $M = 2$

$$Pcs = (\theta^2/(\theta+1)^3)(\theta+3), \quad L=3 \quad (8.a)$$

$$Pcs = (\theta^2/(\theta+1)^4)(\theta^2+4\theta+3), \quad L=4 \quad (8.b)$$

$$Pcs = (\theta^3/(\theta+1)^5)(\theta^2+5\theta+10), \quad L=5 \quad (8.c)$$

For $M = 4$

$$Pcs = (\theta/(\theta+3)^3)(\theta^2+9\theta+6), \quad L=3 \quad (9.a)$$

$$Pcs = (\theta/(\theta+3)^4)(\theta^3+12\theta^2+45\theta+6), \quad L=4 \quad (9.b)$$

$$Pcs = (\theta^2/(\theta+3)^5)(\theta^3+15\theta^2+90\theta+150), \quad L=5 \quad (9.c)$$

For $M = 8$

$$Pcs = (\theta/(\theta+7)^3)(\theta^2+21\theta+42), \quad L=3 \quad (10.a)$$

$$Pcs = (\theta/(\theta+7)^4)(\theta^3+28\theta^2+273\theta+210), L=4 \quad (10.b)$$

$$Pcs = (\theta/(\theta+7)^5)(\theta^4+35\theta^3+490\theta^2+2145\theta+840), L=5 \quad (10.c)$$

$Pcs$ is evaluated for each combination of $M$ and $L$ in order to find the probability of error, which equals $(1-Pcs)$. This method is applicable only when the average SNR of the diversity channels are equal. When average SNR of each channel is different, a simulation is done to find the probability of error. The simulation will count the number of votes in different cells, for a particular hypothesis, and increment error count after each iteration, if the correct hypothesis doesn't get the maximum number of counts. The ties are broken by randomization, i.e., if number of ties equal to three, including the correct one, then probability of picking up the correct one is 1/3 and so on.

## III. Square Law Combiner

Though the complex MRC (maximum ratio combiner) is the optimal combiner when channel phase is known, for non coherent detection, square law combiner is optimal when all the relay links are independent and identically distributed as Rayleigh (see Appendix for a proof). The output of the combiner containing the signal (assumed as $U_1$ without any loss of generality) is [8],

$$U_1 = \sum_{k=1}^{L} | 2\zeta\alpha_k e^{-j\phi_k} + N_{k1} |^2 \quad (11)$$

where $\{\alpha_k e^{-j\varphi_k}\},\{N_{k1}\}$, are complex valued zero mean Gaussian random variable.While the output of the remaining $M$-1 combiners are :

$$U_m = \sum_{k=1}^{L} | N_{km} |^2, m = 2,3,4....M \quad (12)$$

Proakis [8] did a detailed analysis of the output of the combiner showing that $U_1$ will have a Chi-square probability density function with $2L$ degrees of freedom, when all diversity channels have equal SNR

$$p(U_1) = \frac{1}{(2\sigma_1^2)^L (L-1)!} U_1^{L-1} \exp\left(-\frac{U_1}{2\sigma_1^2}\right) \quad (13)$$

where,

$$\sigma_1^2 = \frac{1}{2} E\left(| 2\zeta\alpha_k e^{-j\varphi_k} + N_{k1} |^2\right) = 2\zeta N_0\left(1+\overline{\gamma_c}\right)$$

and $\overline{\gamma_c}$ is the average SNR per diversity channel.

The output of the other combiners, $U_2,\ldots\ldots\ldots,U_M$ are identically distributed with the probability density function given by

$$p(U_2) = \frac{1}{(2\sigma_2^2)(L-1)!} U_2^{L-1} \exp\left(-\frac{U_2}{2\sigma_2^2}\right) \quad (14)$$

where,

$$\sigma_2^2 = 2\zeta N_0.$$

The probability of error is simply 1 minus the probability that $\bigcap_{m=2}^{M} U_1 > U_m$.

Now,

$$P(U_2 < U_1) = \int_0^{U_1} p(U_2)\,dU_2$$

$$= 1 - \exp\left(\frac{-U_1}{2\sigma_2^2}\right)\sum_{k=0}^{L-1}\frac{1}{k!}\left(\frac{U_1}{2\sigma_2^2}\right)^k \quad (15)$$

With $U_1$ fixed, the joint probability $P(U_2 < U_1, U_3 < U_1, \ldots\ldots U_m < U_1)$ is equal to $P(U_2 < U_1)$ raised to the $(M-1)$th power. The $(M-1)$th power of this probability is then averaged over the probability density function of $U_1$ to yield the probability of correct decision. If this result is subtracted from unity then the probability of symbol error could be written in the following form

$$P_M = 1 - \int_0^{\infty}\left[\left\{\frac{1}{(2\sigma_1^2)^L (L-1)!} U_1^{L-1} \exp\left(\frac{-U_1}{2\sigma}\right)\right\}\times\right.$$

$$\left.\left\{1 - \exp\left(\frac{U_1}{2\sigma_2^2}\right)\sum_{k=0}^{L-1}\frac{1}{k!}\left(\frac{U_1}{2\sigma_2^2}\right)^k\right\}^{M-1}\right] dU_1 \quad (16)$$

Using Binomial expansion for the ($M$-1)th power transform, $P_M$ could be expressed as

$$P_M = \left[\frac{1}{(L-1)!}\sum_{m=1}^{M-1}\frac{(-1)^{m+1}\binom{M-1}{m}}{(1+m+m\overline{\gamma_c})^L}\right.$$

$$\left.\times\sum_{k=0}^{m(L-1)}\beta_{km}(L-1+k)!\left(\frac{1+\overline{\gamma_c}}{1+m+m\overline{\gamma_c}}\right)^k\right] \quad (17)$$

With no diversity ($L$=1), the error probability reduces to the simple form

$$P_M = \sum_{m=1}^{M-1}\frac{(-1)^{m+1}\binom{M-1}{m}}{1+m+m\overline{\gamma_c}} \quad (18)$$

To apply the result stated in (17) we need to calculate the coefficient $\beta_{km}$ for different sets of $M$ and $L$, using the following equation

$$\left(\sum_{k=0}^{L-1}\frac{U_1^k}{k!}\right)^m = \sum_{k=0}^{m(L-1)}\beta_{km}U_1^k \quad (19)$$

Now we will concentrate for set of diversity channels that have different SNR. If $\gamma_b$ is the sum of $L$ statistically independent components $\gamma_k$, which is the instantaneous SNR of $k^{th}$ channel, the probability density function of $\gamma_b$ can be written as,

$$p(\gamma_b) = \sum_{k=1}^{L}\frac{\pi_k}{\overline{\gamma_k}} e^{\frac{-\gamma_b}{\overline{\gamma_k}}} \quad (20)$$

where $\pi_k$ is defined as,

$$\pi_k = \prod_{\substack{i=1\\ i\neq k}}^{L}\frac{\overline{\gamma_k}}{\overline{\gamma_k}-\overline{\gamma_i}}$$

and $\overline{\gamma_k}$ is the average SNR of the $k^{th}$ channel.

The probability of symbol error, conditioned on a specific $\gamma_b$, could be expressed as

$$P_M = \sum_{n=1}^{M-1}(-1)^{n+1}\binom{M-1}{n}\frac{1}{n+1}\exp\left[\frac{-nc\gamma_b}{(n+1)}\right] \quad (21)$$

where $c = \log_2 M$. By unconditioning $P_M$ with respect to probability density function of $\gamma_b$, we get the final probability of error

$$P_e = \int_0^{\infty}\left(\sum_{n=1}^{M-1}(-1)^{n+1}\binom{M-1}{n}\frac{1}{n+1}\exp\left[\frac{-nc\gamma_b}{(n+1)}\right]\right)p(\gamma_b)\,d\gamma_b \quad (22)$$

Therefore,

$$P_e = \sum_{n=1}^{M-1}(-1)^{n+1}\binom{M-1}{n}\sum_{k=1}^{L}\frac{1}{1+n+nc\overline{\gamma_k}}\prod_{\substack{i=1\\ i\neq k}}^{L}\frac{\overline{\gamma_k}}{\overline{\gamma_k}-\overline{\gamma_i}} \quad (23)$$

## IV. Numerical Results

Two cases of channels having equal SNR and channels having unequal SNR are considered. The counting rule method is compared with the results obtained from square law combiner. Number of hypothesis ($M$) for which this numerical evaluations are done are two, four or eight. Number of diversity channels ($L$) are three, four or five.

### A. Channels with equal SNR

Graphs are plotted for same $M$ and different $L$ in the same figure, so they can be compared with respect to diversity. We have applied theoretical results obtained from equations (8), (9), (10) for counting rule and (17) for square law combiner. For square law combiner we needed to calculate the coefficients $\beta_{km}$, for each set of $M$ and $L$ using equation (19), which becomes tedious for $M$ = 8, and hence $M$ = 8 is omitted from the calculations. Average channel SNR values are assumed to range from 6 dB to 16 dB. Fig. 1 and Fig. 2 show the probability of error comparison of the two methods. For all cases, the probabilities of error using counting rule are higher than those achieved using square law combiner, although not significantly large when average channel SNR is low. Also, fewer number of hypothesis gives lower error rate, which is true both for counting rule and square law. As the number of relays increases, the probability of error decreases irrespective of the number of hypotheses , both in counting rule and square law.

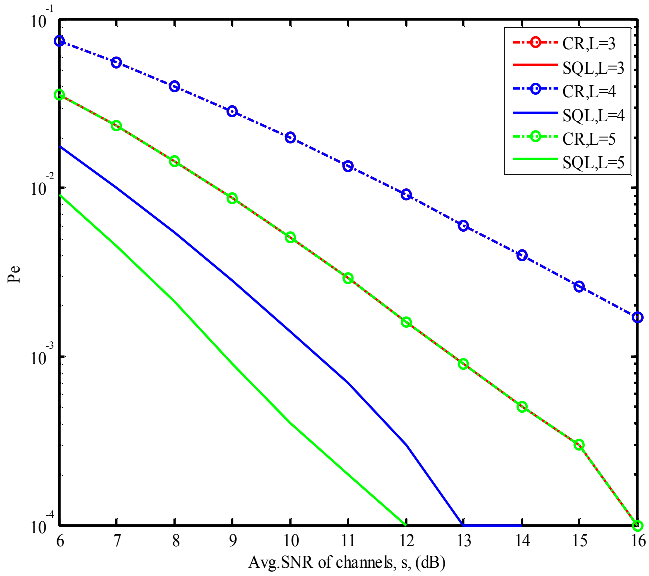


Fig. 1 Plot of probability of error vs. SNR of channels for $M$=2

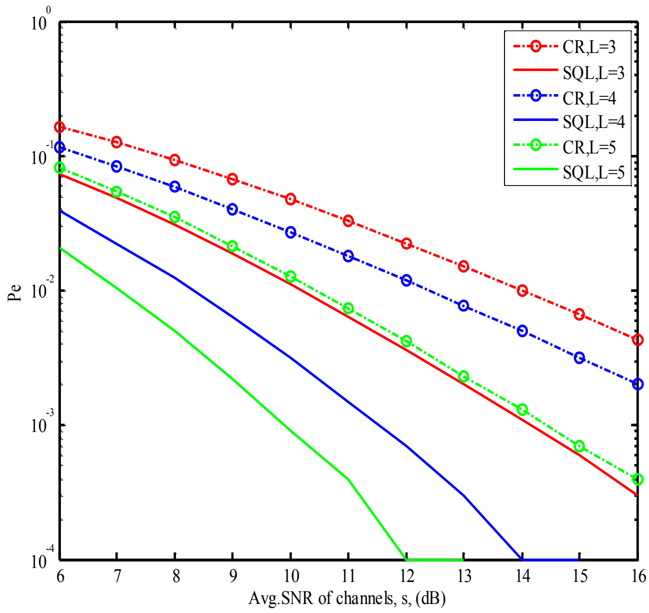


Fig. 2 Plot of probability of error vs. SNR of channels for $M$=4

## B. Channels with unequal SNR

To calculate probability of error for channels having unequal SNR, instead of theoretical approach simulation is done in counting rule method. The channel SNRs are assumed to be,

$$[s-2 \quad s \quad s+2], \qquad L=3$$

$$[s-3 \quad s-1 \quad s+1 \quad s+3], \qquad L=4$$

$$[s-4 \quad s-2 \quad s \quad s+2 \quad s+4], \qquad L=5$$

where $s$ is the average of the SNRs of all channels, which ranges from 6 dB to 16 dB. The spacing between SNR of adjacent channels is 2 dB. Another set of calculations are done for channels spaced 4 dB apart.

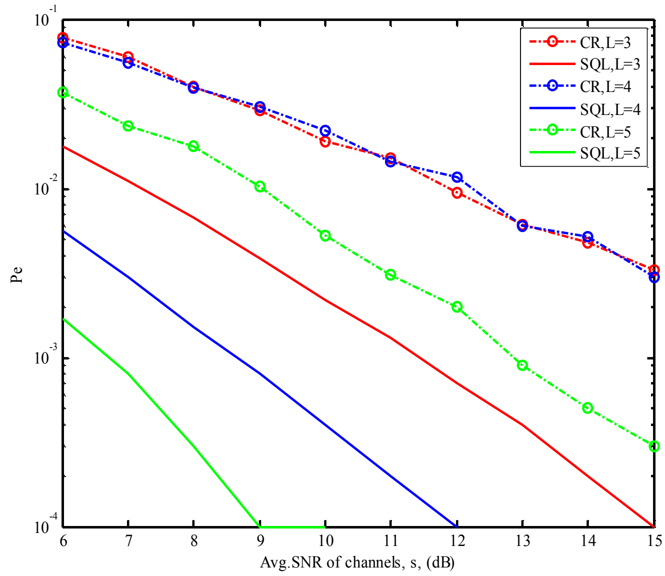


Fig. 3 Plot of probability error vs. SNR for $M=2$ , and unequal relay SNRs are 2dB apart

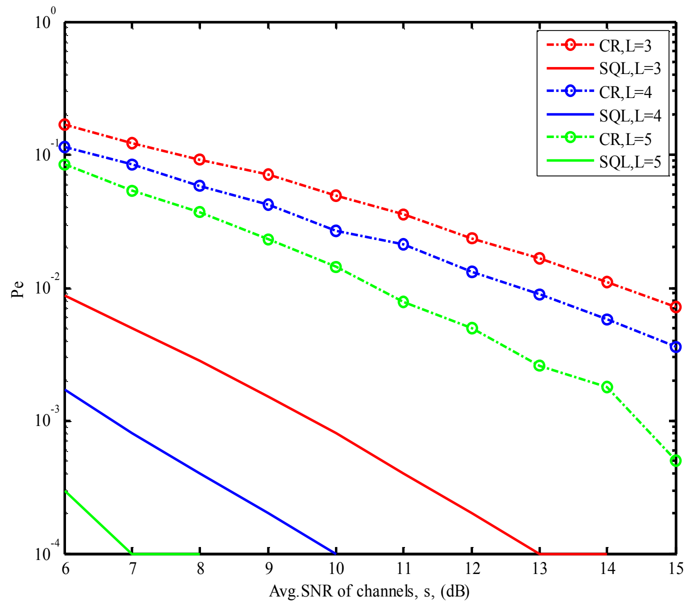


Fig. 4 Plot of probability error vs. SNR for $M=4$, and unequal relay SNRs are 2dB apart

Fig. 3 through Fig. 8 show comparison of error performances of counting rule and square law.
Clearly, for the unequal average SNR case, the square law provides significantly smaller probability of error than the counting rule, for all SNRs, including the small values.
Fig. 3 and Fig. 6 show that for $M$=2, counting rule gives almost identical probabilities of errors for both $L$=3 and $L$= 4, as in the case of equal SNR channel. However, the

results obtained for $L$=3 and $L$=4 using square law are quite different. Square law performs much better than counting rule for $L$=4 and $L$=5.

## V. Conclusion

In this paper we compared two combining procedures, counting rule and square law, for $M$-ary FSK detection in relay networks. The network consists of a single source - destination pair and $L$ number of relays. The transmission channel is assumed to be slow Rayleigh fading channel and signals are corrupted by AWGN. As expected, the square law performs better for all cases. For equal average channel SNRs, error rates are not too far apart when the diversity channel has small average SNR. Simplicity of counting rule may be advantageous in such cases. Counting rule performs much inferior to square law combiner for unequal relay channel average SNRs as compared to the equal SNR channel.

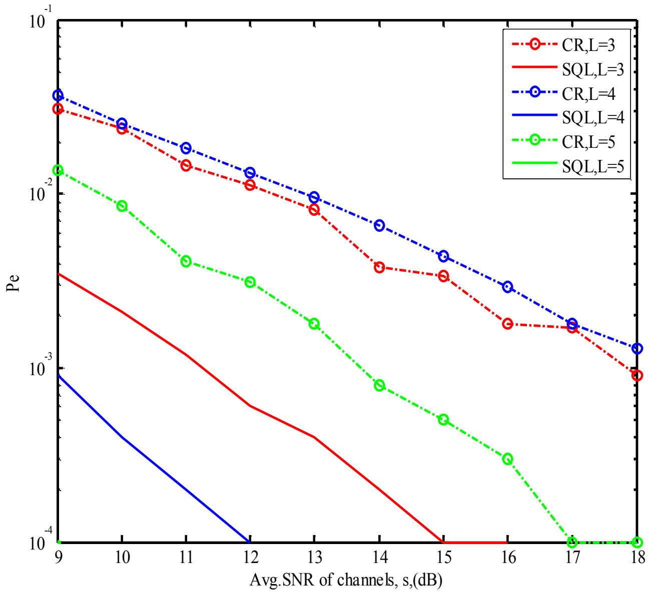


Fig. 6 Plot of probability of error vs. SNR for $M$ = 2, unequal relay SNRs are 4dB apart

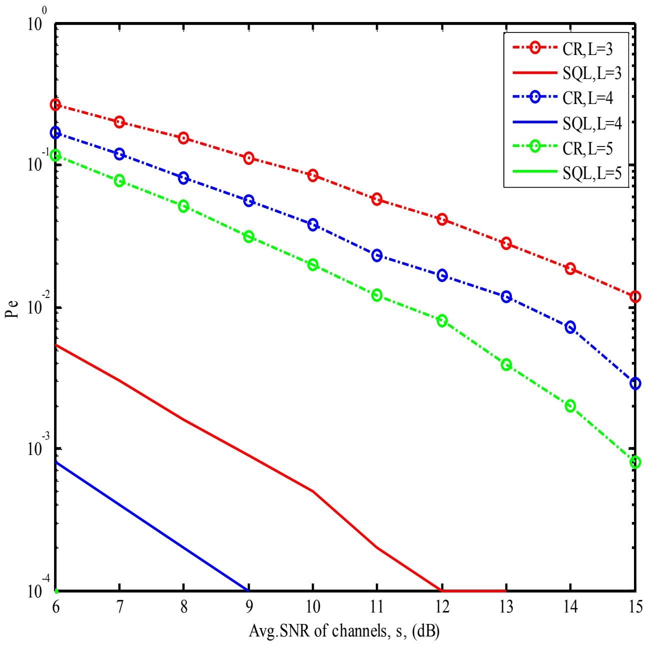


Fig. 5 Plot of probability of error vs. SNR for $M$ = 8, unequal relay SNRs are 2dB apart

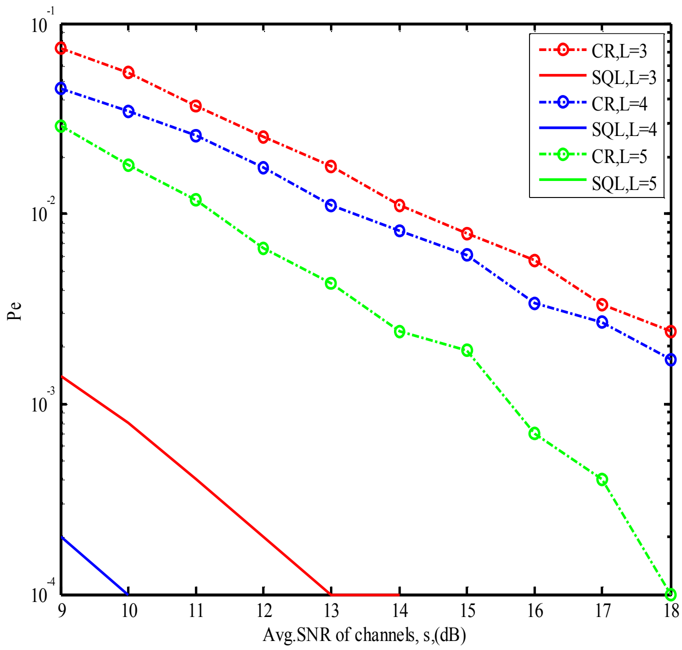


Fig. 7 Plot of probability of error vs. SNR for $M$ = 4, unequal relay SNRs are 4dB apart

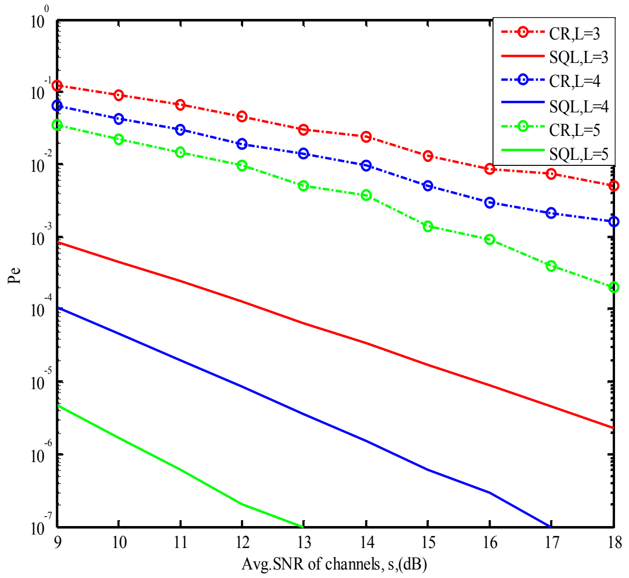


Fig. 8 Plot of probability error vs. SNR for $M = 8$, unequal relay SNRs are 4dB apart

# VI. References


[1] Theodore S. Rappaport, *Wireless Communications : Principles and Practice*, Prentice Hall, 2nd Edition, 2002.

[2] J. Nicholas Laneman, *Ph.D. Dissertation*, MIT, Department of Electrical Engineering and Computer Science, 2002.

[3] Andrew Sendonaris, Elza Erkip and Behnaam Aazhang, "User cooperation diversity-part I : System description ", pp. 1927-1938, "User cooperation diversity – Part II : Implementation aspects and performance analysis" *IEEE Transactions on Communication*, Vol. 51, No. 11, pp. 1939-1948, November 2003.

[4] Bin Liu, Biao Chen, Rick S. Blum, "Minimum error probability cooperaitve relay design", *IEEE Transaction on Signal Processing*, Vol. 55, No. 2, pp. 656-664, February 2007.

[5] Naveen Shastry, Raviraj S. Adve, " A theoretical analysis of cooperative diversity in wireless sensor networks", *IEEE GLOBECOM*, pp. 3269-3273, 2005.

[6] Chirag S. Patel, Gordon L. Stuber and Thomas G. Pratt, "Statistical properties of amplify and forward relay fading channels", *IEEE Transactions on Vehicular Technology*, Vol. 55, No. 1, pp. 1-9, January 2006

[7] R. Niu, B. Chen, and P. K. Varshney, "Decision fusion rules in wireless sensor networks using fading channel statistics", in *IEEE Transaction on Signal Processing*, Vol. 54, No. 3, pp. 1018-1027, March, 2006.

[8] John G. Proakis, *Digital Communication*, Mc Graw Hill, 4th edition, 2001.

[9] Shanti S. Gupta, S. Panchapakesan, *Multiple Decision Procedure*, SIAM, Philadelphia, 2002


# VII. APPENDIX

Let $U_m$ denote the square law output from the $m$th frequency filter in the $l$th relay.

$$U_m = \sum_{l=1}^{L} U_{ml}, m = 1,2,.., M \qquad \text{(i )}$$

Assuming $U_k$ contains the signal, the maximum likelihood test picks

$$\max_k \quad p(U_{11},..,U_{1L},U_{21},..,U_{2L},...,U_{M1},...,U_{ML} \mid H_k)$$

$$= \max_k \left( (\frac{1}{2\sigma_1^2}\exp(-\sum_{l=1}^{L}\frac{U_{kl}}{2\sigma_1^2}) \prod_{\substack{j=1\\j\neq k}}^{M} \frac{1}{2\sigma_2^2}\exp(-\frac{1}{2\sigma_2^2}\sum_{l=1}^{L} U_{jL}) \right)$$

$$= \min_k \left( \sum_{l=1}^{L} \frac{U_{kl}}{2\sigma_1^2} + \sum_{\substack{j=1\\j\neq k}}^{M} \frac{\sum_{l=1}^{L} U_{jl}}{2\sigma_2^2} \right) \qquad \text{(ii)}$$

If we define $\theta$ as

$\theta = SNR + 1$, and

$U_j = \sum_{l=1}^{L} U_{jl}$ , then equation (ii) becomes

$$= \min\left( \frac{1}{\theta}U_1 + \sum_{\substack{j=1\\j\neq 1}}^{M} U_j, \frac{1}{\theta}U_2 + \sum_{\substack{j=1\\j\neq 2}}^{M} U_j, ..., \frac{1}{\theta}U_M + \sum_{\substack{j=1\\j\neq M}}^{M} U_j \right) \qquad \text{(iii)}$$

Let us suppose that $k$th term is the minimum. In order that the receiver picks correctly the $k$th hypothesis

$$\frac{1}{\theta}U_k + \sum_{\substack{j=1\\j\neq k}}^{M} U_j < \frac{1}{\theta} + \sum_{\substack{j=1\\j\neq 1}}^{M} U_j$$

$$\frac{1}{\theta}U_k + \sum_{\substack{j=1\\j\neq k}}^{M} U_j < \frac{1}{\theta} + \sum_{\substack{j=1\\j\neq 2}}^{M} U_j$$

$$\vdots$$

$$\frac{1}{\theta}U_k + \sum_{\substack{j=1\\j\neq k}}^{M} U_j < \frac{1}{\theta} + \sum_{j=1}^{M-1} U_j$$

Or

$$U_1(1-\frac{1}{\theta}) < U_k(1-\frac{1}{\theta})$$

$$U_2(1-\frac{1}{\theta}) < U_k(1-\frac{1}{\theta})$$

$$\vdots$$

$$U_M(1-\frac{1}{\theta}) < U_k(1-\frac{1}{\theta})$$

Since $\theta > 1$, i.e., $\frac{1}{\theta} < 1$

$U_k > U_1$, $U_k > U_2$,.., $U_k > U_M$.

i.e., $U_k$ is the maximum among $(U_1, U_2 .., ... U_M)$.